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Dent patterns on the surface of a longitudinally compressed, non-linear elastic cylindrical shell☆

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Abstract

A novel version of reductive perturbation theory is proposed for analysing the dynamics of bends in a longitudinally compressed, non-linear elastic cylindrical shell near the stability threshold given by the linear theory. Soliton-like annular folds and patterns of diamond-shaped dents on the shell surface are predicted and analytically described. Similar formations, which are both stress concentrators and precursors of plastic flow of the material, contain information on the precritical stress state of the shell. It is shown that a shell with dents supports an external load, which is tens of percent less than the upper critical load in the linear theory of shells. The conditions for the formation of and explicit expressions for solitary waves that propagate along the generatrix of the shell on a background of arrays of folds and dents are found.

The linear theory of shells[1,2] starts from the assumptions that the displacements of point masses in a shell are small compared with its thickness and that the strains obey Hooke's law. These assumptions narrow the range of applicability of this theory. The criterion for instability of the shape of a shell in the linear theory is the formation of neutrally stable deformation modes. The minimum strain at which neutrally stable modes form corresponds to the critical stress. Near the stability threshold of the system, the amplitude of the bends in the shell begins to increase. At this point the approximations of the linear theory cease to be valid, and it gives overestimated values for the stresses at which the shell loses shape stability. In practice, after the loss of shape stability at the "upper" critical load specified by the linear theory, the shell "snaps," and spatially localized bends and "dent patterns" form on its surface. The limiting load that can be supported by a shell with dents falls sharply and becomes less sensitive to changes in various random factors, such as imperfection of the initial shape of the shell or stress non-uniformity. Since the deflections of the shell are comparable with its thickness, the process of spatial localization of the bends on the shell surface cannot be described by the linear theory. At the same time, the strains of the middle surface of a thin shell with dents are usually small and correspond mainly to geometric bending;[3] therefore, the initial stage of deformation of a shell can be investigated within the non-linear theory of elasticity.[4–6] In this stage the non-linear interactions of close unstable deformation modes and dispersion effects begin to compete with one another, leading to a restriction on the increase in the amplitude of the waves and the formation of patterns of spatially localized bends on the surface of the shell near its stability threshold.

A constructive solution of the problem of the dynamics of a shell near its stability threshold is possible within simplified models that correctly take into account the principal interactions and, at the same time, allow of exact

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solutions. One of the important features of the problem is the fact that the original equations of the non-linear theory of elasticity for an infinite medium do not contain dispersion terms. When simplified models are constructed, such terms appear as a result of the elimination of the "fast" variables that characterize the inhomogeneity of the strains along a normal to the shell surface. The dispersion has a geometric origin: it depends on the dimensions of the shell, and the boundary conditions on its developed surface. When there are no dispersion effects, when only the interaction of unstable deformation modes is taken into account, dent localization on the shell surface is impossible. Therefore, boundary-value problems must be solved accurately along a normal to the shell surface.

We recall that there is a similarity between thin plates and shells that allows us to use some techniques of the theory of plates to construct a theory of shells. However, the conventional approximations, which are based on Kirchhoff's geometric hypotheses or expansions of displacements in Taylor series along the coordinate perpendicular to the plate surface, often do not satisfy the boundary conditions on the plate surface. Such techniques cannot be developed into an exact theory. This deficiency was first noted by Novozhilov et al.[1,7], and it was found that the error of Kirchhoff's hypotheses is more significant even in the linear theory of shells than in the theory of plates. It was shown in Ref. 8 in the case of an analytic description of two-dimensional multisolitons in a plate that hypotheses and formulations that are typical of the variational approach do not give a correct dispersion law of linear modes and lead to incorrect estimates and qualitative conclusions when the dynamics of non-linear elastic systems are analysed.

The non-linear theory of elasticity, unlike the classical non-linear theory of shells,[9,10] takes into account not only the geometric non-linearity of the medium, which manifests itself in the non-linearity of the strain tensor, but also its physical non-linearity, which characterizes the properties of the material and is described by high-order invariants in the expansion of the elastic energy of the medium.

The fundamental equations of the traditional non-linear theory of shells represent the equilibrium conditions of elements of the middle surface of a shell. It is difficult to take into account the effects of non-linear dispersion and the higher anharmonicities of local strains of a shell within such an approach.

The equations of the non-linear theory of elasticity are usually simplified by making additional assumptions regarding the role of the invariants in the expansion of the energy of a non-linear elastic body: one of them is retained, while others of the same order of magnitude are discarded. In a systematic description of the non-linear elastic dynamics of shells, it is more correct to start from a complete expansion of the elastic energy of the material in invariants of the strain tensor that is consistent with the symmetry of the medium. The development of the physics of non-linear phenomena has resulted in the appearance of methods for constructing simplified models,[11] which enable the non-linear elastic dynamics of a shell near its stability threshold to be approximated correctly without enlisting a priori hypotheses and have verifiable accuracy with respect to the small parameters that characterize the dimensions of the shell, the space-time scales of the strains, the geometric and physical non-linearity of the material and the magnitude of the external stress. It is important that the procedure for constructing a model can disclose the hidden dynamic symmetry of the problem; therefore, the simplified equations are universal and correspond closely to the integrable non-linear models, opening up the possibility of a detailed analysis of their solutions by the methods of modern soliton theory.

The non-linear elastic dynamics associated with the formation of "patterns of folds and dents" on shell surfaces and the possibility of approximating them by integrable models have scarcely been studied.

In this paper we propose a version of reductive perturbation theory that is suitable for solving non-linear boundary-value problems in which the final shape of the surface of a deformable shell is not known a priori and is found when solving the problems. The initial non-linear elastic energy of the medium contains all the invariants of the strain tensor that are compatible with the symmetry of the medium. The proposed method automatically selects the invariants of the strain tensor in the equations of the non-linear theory of elasticity that are needed to solve a specific problem. The method is used to construct a simplified model of the dynamics of axisymmetric bends on the surface of a longitudinally compressed, non-linear elastic shell near the stability threshold predicted by the linear theory. It is important that in this problem the development of local instabilities is caused not only by the geometric non-linearity of the medium, but also by its physical non-linearity and, therefore, cannot be investigated within the conventional theory of shells.

The case of the loading of a shell considered here is of practical interest, and many problems can be reduced to it. A circular shell compressed along its generatrix not only serves as the principal load-bearing element of many structures, but is also a standard for comparing theoretical and experimental data and testing different approaches in the theory of the stability of shells. This study shows that the non-linear theory of elasticity can be used to describe analytically the formation of spatially localized annular folds on the surface of a longitudinally compressed shell. The localization of folds is associated with non-linear interactions that are not taken into account by the conventional models of shells in the "first and second" approximations.[9]

When a circular cylindrical shell is subjected to uniaxial compression, the linear theory of elasticity predicts the formation of a checkerboard array of dents and bulges of rectangular shape on the shell surface in addition to the axisymmetric bending of the shell. However, instead of such a structure, two-dimensional spatially localized patterns of diamond-shaped dents are usually observed on the shell surface.[2] The reductive perturbation theory procedure proposed in this paper allows of generalization and provides an analytic description of the two-dimensional patterns of diamond-shaped dents on the surface of a longitudinally compressed shell. To construct the corresponding model, a hierarchy of the following variables for describing the various scale levels of the strains in the shell is introduced: 1) "fast" variables, that characterize strains along a normal to the shell surface, 2) "slower" variables, that are responsible for the formation of dents on its surface, and 3) variables that specify the localization of groups of dents on the shell surface. When constructing the model, a series of non-linear boundary-value problems is solved along a normal to the surface of the strained shell. An important role is played by the conditions for solvability of the boundary-value problems by reductive perturbation theory. They give: a) the wave vectors of the neutrally stable resonant deformation modes that cause the formation of diamond-shaped dents on the shell surface, b) the critical external stress at which the resonant modes form, and c) a simplified model of shell dynamics. The model specifies the evolution of the envelopes of the diamond-shaped dents on the shell surface. The boundary conditions for the end surfaces of the shell are replaced by effective boundary conditions for the simplified non-linear model.

At first glance, it appears that two neutrally stable deformation modes should be responsible for the formation of the diamond-shaped dents. In particular, this was assumed in the linear theory when calculating the "checkerboard array" of bulges and depressions on the shell surface. However, a more detailed analysis reveals that the formation of two-dimensional dent patterns is due to the interaction of three groups of unstable waves that are near three neutrally stable deformation modes.

It is shown that patterns of stationary diamond-shaped dents form on shell surfaces at loads that are tens of percent less than the "upper" critical load of the linear theory. A shell with diamond-shaped dents on its surface supports external stresses over a limited range. They are above a certain critical value, but below the upper critical load of the linear theory.

Since shells with folds or dents maintain their shape only over a certain range of loads, their spatially localized, nonlinear elastic strains may be considered as stress concentrators, i.e., "precursors" of the plastic flow of the material, which occurs when the external load is increased further.

In addition, the models enable us to find the conditions for the formation of and explicit expressions for solitary waves that propagate along the generatrix of a shell on a background of arrays of annular folds and diamond-shaped dents. Since solitary waves form only in the initial stage of shell deformation, their analysis may be useful for revealing and diagnosing the pre-critical stress state of a shell.

A comparison of two closely related problems (the description of annular folds and of diamond-shaped dents) shows that different invariants of the strain tensor are important for a theoretical description of different scenarios for the deformation of a longitudinally compressed shell. The proposed non-linear perturbation theory scheme can be used to analyze any scenario.

The procedure for constructing effective models based on equations of the non-linear theory of elasticity is tedious. However, it is universal and enables the non-linear elastic dynamics of real complex systems near their stability threshold to be approximated accurately by simple models that allow of a detailed analysis using techniques from soliton theory and numerical methods. The model equations are suitable for studying the evolution of the shape of a shell after it loses stability, as long as the strains remain non-linear elastic and comparatively small.

1. Fundamental equations

Suppose the axis of a circular cylindrical shell coincides with the Ox_1 axis of a Cartesian system of coordinates, and let $\mathbf{r} = x^s \mathbf{i}_s$ be the radius vector of a material particle of the unstrained body, where $\mathbf{i}_s$ denotes the unit vectors of the Cartesian system of coordinates and summation is performed over repeated indices. The indices that are denoted by Latin letters take the values 1, 2, 3.

We will change from the Cartesian system of coordinates to a cylindrical system of coordinates because it more closely reflects the symmetry of the shell:

$$y^1 = x^1, \quad y^2 = \frac{R}{2i}\ln\frac{x^2 + ix^3}{x^2 - ix^3}, \quad y^3 = [(x^2)^2 + (x^3)^2]^{1/2}$$

Here R is the radius of the middle surface of the shell, d is the thickness of the shell and $|y^3 - R| \le d/2$.

The basis vectors of a local reference point in the cylindrical system of coordinates and the metric defining the distance $(d\mathbf{r})^2 = g_{ik}dy^i dy^k$ between two close material particles of the unstrained body have the form

$$\mathbf{e}_i = \frac{\partial x^s}{\partial y^i}\mathbf{i}_s, \quad g_{ik} = \frac{\partial x^s}{\partial y^i}\frac{\partial x^s}{\partial y^k} = \operatorname{diag}\left\{1, \left(\frac{y^3}{R}\right)^2, 1\right\}$$

When the body is deformed, the radius vector of each material particle in it undergoes a displacement $\mathbf{u}(\mathbf{r}, t)$: $\mathbf{R} = \mathbf{r} + \mathbf{u}(\mathbf{r}, t)$, which can be represented in the form $\mathbf{u} = v^s\mathbf{e}_s$. Here v^s denotes the coordinates of the displacement vector of a material particle of the medium relative to a local reference point associated with the unstrained body.

The Lagrange strain tensor E_{km}, which characterizes the change in the distance between two close material particles when the body is deformed, is defined by the relation

$$E_{km} = \frac{1}{2}\{\nabla_k v^s g_{sm} + \nabla_m v^s g_{ks} + \nabla_k v^s \nabla_m v^p g_{sp}\}; \quad \nabla_k v^s = \frac{\partial}{\partial y^k}v^s + \Gamma^s_{kp}v^p$$

Here $\nabla_k v^s$ is the absolute (covariant) derivative of the components v^s of the displacement vector, which takes into account the changes in the length and orientation of the vectors of the local reference point on transferring from one point in the medium to a neighbouring point. In the present case, only the following Christoffel symbols, which specify the operation of parallel transposition in a curvilinear system of coordinates, are non-zero:

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{y^3}, \quad \Gamma^3_{22} = -\frac{y^3}{R^2}$$

In a curvilinear system of coordinates, the superscripts and subscripts on the tensors and absolute derivatives play different roles. They are raised and lowered using the metric tensor g_{ik} and its inverse:

$$g^{ks} = (\partial y^k/\partial x^i)(\partial y^s/\partial x^i) = \operatorname{diag}\{1, (R/y^3)^2, 1\}$$

The non-linear elastic energy of the material is written down in the form of an expansion in invariants of the strain tensor that take into account the crystallographic symmetry of the medium.[4–6] For an isotropic medium, there are three independent invariants

$$I_1 = E^m_m; \quad I_2 = E^p_m E^m_p; \quad I_3 = E^p_m E^m_k E^k_p$$

The general expression for the elastic energy of an isotropic non-linear body has the form[4–6]

$$U = \int_{V_0} \varphi\sqrt{g}\,dy^1dy^2dy^3$$

$$\varphi = \frac{\lambda}{2}I_1^2 + \mu I_2 + \frac{A}{3}I_3 + BI_1I_2 + \frac{C}{3}I_1^3 + \varphi_{n\ge4}, \quad \varphi_{n\ge4} = \sum_{n=4}^{\infty}\sum_{\langle k,p,q\rangle=n} A_{kpq}I_1^kI_2^pI_3^q$$

$$\sqrt{g} = \sqrt{\det\|\mathbf{g}\|} = \det\left\|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right\| = -\frac{y^3}{R} \tag{1.1}$$

Here φ is the energy per unit volume of the body before deformation, $\sum_{\langle k,p,q\rangle=n}$ denotes the sum of the terms for which $k+2p+3q=n$, and integration is performed over the volume of the body V_0 before deformation. The moduli of elasticity λ, μ, A, B, C, ... are assumed to be comparable in order of magnitude. The initial energy φ contains all the invariants of the strain tensor that are compatible with the symmetry of the medium. It is noteworthy that when constructing the simplified non-linear model, the mathematical algorithm of reductive perturbation theory correctly picks out both the invariants and the contributions from them that are needed to solve a specific problem.

The equations of the dynamics of a non-linear elastic body have the form[4–6]

$$-\rho_0\frac{\partial^2}{\partial t^2}\upsilon^i+\nabla_s P^{is} = 0 \tag{1.2}$$

where $\rho_0=\text{const}$ is the density of the medium in the unstrained state and P^{is} denotes the components of the Piola–Kirchhoff tensor that is asymmetric with respect to the indices i and s:

$$P^{is} = \frac{\partial\varphi}{\partial E_{si}}+\frac{\partial\varphi}{\partial E_{sm}}\nabla_m\upsilon^i \tag{1.3}$$

In the non-linear theory of elasticity, the forces that appear during deformation are characterized by a symmetric stress tensor, which represents the momentum flux density (Θ^{mn} are the components of the stress tensor relative to a local reference point). The Piola–Kirchhoff tensor is related to the stress tensor as follows:

$$\Theta^{mn} = \frac{1}{\det\|\mathbf{C}\|}P^{mk}C_k^n = \frac{1}{\det\|\mathbf{C}\|}C_k^m P^{nk},\quad C_m^s = \delta_m^s+\nabla_m\upsilon^s$$

On the part of the surface of the strained sample where the surface forces $\mathbf{f}=f^k\mathbf{e}_k$ are assigned, the following boundary conditions are satisfied (N_i denotes the components of the vector of a unit normal to the oriented area dS)

$$\Theta^{ki}N_i\mathrm{d}S = f^k\mathrm{d}S$$

which can be rewritten in terms of the unstrained body[4,5]

$$P^{ki}\nu_i = \Theta^{kn}\frac{\partial\det\|\mathbf{C}\|}{\partial C_i^n}\nu_i = f^k\frac{\mathrm{d}S}{\mathrm{d}\sigma}$$

Here ν_i denotes the components of the vector of the unit normal to the element of area dσ of the body before deformation, and the multiplier dS/dσ characterizes the relative variation of the areas when the body is deformed

$$\frac{\mathrm{d}S}{\mathrm{d}\sigma} = \sqrt{m_k m^k};\quad m_k = \frac{\partial\det\|\mathbf{C}\|}{\partial C_i^k}\nu_i$$

In this paper we will consider a cylindrical shell that is subjected to compression along its generatrix by forces uniformly distributed along its edges. On the unloaded lateral surface of the cylinder σ', the boundary conditions take the simple form

$$P^{i3}\big|_{\sigma'} = 0,\quad i = 1,2,3 \tag{1.4}$$

These non-linear boundary conditions specify the process of localization of dents on the shell surface and are, therefore, accurately taken into account when constructing the model.

It is important that detailed information regarding the distribution of the forces along the shell edges is not needed to construct the equations describing the dynamics of bends located in the central part of the shell. Only the longitudinal stresses that exist in the strained shell outside a narrow strip along its edges appear in the model equations. The problem is greatly simplified by the fact that only the first orders of reductive perturbation theory, for which the approximation $\Theta^{11}\approx P^{11}$ holds, are needed to construct the model. The perturbation-theory procedure corresponds to the Saint-Venant principle, according to which the dynamics of bends in the central part of a shell can determine only the integral characteristics of the forces along the shell edges. The forces acting on the narrow strip are taken into account through the effective boundary conditions for the simplified model precisely as in the conventional theory of shells.[12]

2. The construction of a model of axisymmetric bends in a shell

To simplify Eq. (1.2), we will introduce small parameters that reflect characteristic space-time scales of the strains and the magnitude of the external load.

Let l be the characteristic dimension of the dents on the shell surface, where $l \ll L$ (L is the length of the shell). The bends in the shell are assumed to be sharp, i.e., the amplitudes of the displacement fields are of the order of the thickness of the shell d.

We define the parameters ε and ε_1, which characterize the small orders of magnitude of the amplitudes of the displacements and curvature of the shell:

$$\varepsilon = d/l \ll 1, \quad \varepsilon_1 = d/R = O(\varepsilon^2)$$

Let external stresses be applied only along the shell edges, and let their order of magnitude be as follows:

$$\Theta^{11}/\mu = O(\varepsilon^2) + O(\varepsilon^4)$$

These conditions demarcate the region of the physical parameters of the problem in which the non-linear dynamics of local bends in the shell can be faithfully described within a simpler quasi-one-dimensional model.

We introduce the dimensionless variables

$$\xi_\alpha = \frac{y^\alpha}{l}\ (\alpha = 1, 2), \quad \eta = \frac{y^3 - R}{d}, \quad \tau = \frac{t}{\tau_{ch}}; \quad u = \frac{\upsilon^1}{d}, \quad \upsilon = \frac{\upsilon^2}{d}, \quad w = \frac{\upsilon^3}{d}$$

where $\tau_{ch} = l/\sqrt{\mu/\rho_0}$ is the characteristic deformation time (μ is the shear modulus).

The equations of the dynamics of the shell (1.2) in the dimensionless variables take the form

$$\begin{aligned}
\mu\varepsilon^2\partial_\tau^2 w &= \varepsilon\partial_{\xi_\alpha}P^{3\alpha} + \partial_\eta P^{33} - (1+\varepsilon_1\eta)\varepsilon_1 P^{22} + \frac{\varepsilon_1}{1+\varepsilon_1\eta}P^{33} \\
\mu\varepsilon^2\partial_\tau^2 u &= \varepsilon\partial_{\xi_\alpha}P^{1\alpha} + \partial_\eta P^{13} + \frac{\varepsilon_1}{1+\varepsilon_1\eta}P^{13} \\
\mu\varepsilon^2\partial_\tau^2 \upsilon &= \varepsilon\partial_{\xi_\alpha}P^{2\alpha} + \partial_\eta P^{23} + \frac{\varepsilon_1}{1+\varepsilon_1\eta}(2P^{13} + P^{32}) \\
&(\partial_\tau = \partial/\partial_\tau,\ \partial_{\xi_\alpha} = \partial/\partial\xi_\alpha,\ \partial_\eta = \partial/\partial\eta)
\end{aligned} \tag{2.1}$$

Since the displacement fields in the case of axisymmetric bends in the shell have the form

$$u = u(\xi, \eta, \tau), \quad \upsilon = 0, \quad w = w(\xi, \eta, \tau)$$

the last equality in (2.1) is satisfied automatically.

The main goal of reductive perturbation theory is to reduce the complicated three-dimensional dynamical system (2.1) to a simpler non-linear model by introducing the slow variables X and T. To construct such a model, a solution of (2.1) is sought in the following form

$$\begin{aligned}
w &= w^{(0,0)}(X,T) + [w^{(0,1)}(X,T)\exp(ik\xi) + \text{c.c.}] + \sum_{n=2}^{\infty}\sum_{l=-\infty}^{\infty} w^{(n,l)}(X,T,\eta)\exp(ikl\xi) \\
u &= u^{(0,0)}(X,T) + \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} u^{(n,l)}(X,T,\eta)\exp(ikl\xi)
\end{aligned} \tag{2.2}$$

The integer superscripts n and l specify the order of the term with respect to ε and the multiplicity of the harmonic that describes the bending of the shell surface, respectively. The requirement that the fields u and w are real imposes the constraints

$$[u^{(n,l)}]^* = u^{(n,-l)}, \quad [w^{(n,l)}]^* = w^{(n,-l)} \tag{2.3}$$

Axisymmetric bending of a shell begins with the formation of a neutrally stable linear mode with wave number k_0 under the external stress $\Theta^{11} = \Theta$, which is identical to the critical stress of the linear theory of shells. The values of k_0 and Θ will be found when solving the problem.

For external stresses close to Θ, the non-linear dynamics of the shell are described by unstable modes, whose wave numbers lie in a small vicinity of the critical wave number k_0. The radius of this vicinity depends on how much the external stress differs from the critical stress. We next assume that

$$(\Theta^{11} - \Theta)/\mu = O(\varepsilon^4) \tag{2.4}$$

Then, using the parameter ε we can determine the slow variables

$$X = \varepsilon\xi, \quad T = \varepsilon^2\tau \tag{2.5}$$

that describe the modulation of the fundamental harmonic, which is proportional to $\exp ik_0\xi$, as a result of its interaction with nearby unstable modes. The non-linear interactions of the nearby unstable modes lead to localization of the bends on the shell surface. This clearly distinguishes the non-linear problem from the linear Euler shell instability problem.

Processes that are comparably slow in time, for which the estimate $\partial_\tau w/w = O(\varepsilon^2)$ holds, are considered. More detailed information concerning the initial conditions of the problem is not required to construct the simplified model.

The choice of the scale transformations (2.5) is based on an analysis of the space-time responses of the system to external perturbations and takes into account the possibility of a balance between the dispersion and non-linearity effects (see Refs. 11, 13 and the references cited therein). The correctness of the choice of the slow variables (2.5) will become clear during the further calculations.

The selected form of the solution leads to the following representations

$$E_{sm} = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} E_{sm}^{(n,l)}\exp(ikl\xi), \quad P^{sm} = \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty} (P^{sm})^{(n,l)}\exp(ikl\xi) \tag{2.6}$$

The relation between the coefficients $w^{(n,l)}$, $u^{(n,l)}$ and $E_{km}^{(n,l)}$ is found by substituting (2.2) into the relations

$$E_{11} = \varepsilon\hat{D}u + \frac{\varepsilon^2}{2}[(\hat{D}u)^2 + (\hat{D}w)^2], \quad E_{22} = \varepsilon_1(1+\varepsilon_1\eta)w + \frac{(\varepsilon_1 w)^2}{2}$$

$$E_{13} = \frac{1}{2}\left\{\varepsilon(\hat{D}w) + \frac{a}{d}\partial_\eta u + \frac{a}{d}\varepsilon[(\hat{D}u)\partial_\eta u + (\hat{D}w)\partial_\eta w]\right\}, \quad E_{12} = E_{23} = 0$$

where $D = \partial_\xi + \varepsilon\partial_X$. The expressions for the tensor components $(P^{km})^{(n,l)}$ follow from (1.3). The first components are most simply related to $E_{km}^{(n,l)}$:

$$\begin{aligned}
&(P^{13})^{(n,l)} = (\partial\varphi/\partial E_{13})^{(n,l)} = 2\mu E_{13}^{(n,l)}, \quad n = 1,2,3,4\\
&(P^{\alpha\beta})^{(s,l)} = (\partial\varphi/\partial E_{\alpha\beta})^{(s,l)} = \lambda E_{pp}^{(s,l)}\delta_{\alpha\beta} + 2\mu E_{\alpha\beta}^{(s,l)}, \quad \alpha,\beta = 1,2\\
&(P^{33})^{(s,l)} = (\partial\varphi/\partial E_{33})^{(s,l)} = \lambda E_{pp}^{(s,l)} + 2\mu E_{33}^{(s,l)}, \quad s = 2,3
\end{aligned} \tag{2.7}$$

After substituting expansions (2.6) into (2.1) and equating terms of the same order of smallness, we obtain a series of ordinary differential equations in the fast variable η, which specifies the non-uniformity of the strains along a normal to the shell surface. The boundary conditions needed to solve them are obtained by expanding the original boundary conditions (1.4) in powers of the small parameter ε.

In the first orders of perturbation theory, the boundary-value problems

$$\partial_\eta (P^{13})^{(n,l)} = 0, \quad (P^{13})^{(n,l)}\big|_{\eta=\pm 1/2} = 0, \quad n = 1, 2 \tag{2.8}$$

$$\partial_\eta (P^{33})^{(m,l)} = 0, \quad (P^{33})^{(m,l)}\big|_{\eta=\pm 1/2} = 0, \quad m = 2, 3 \tag{2.9}$$

have trivial solutions. The solution of (2.8) is equivalent to the equalities

$$(P^{13})^{(n,l)} = (\partial\varphi/\partial E_{13})^{(n,l)} = 2\mu E_{13}^{(n,l)} = 0, \quad n = 1, 2$$

from which it follows first and foremost that the coefficients $(P^{13})^{(n,l)}$ and $E_{13}^{(n,l)}$ are actually not equal to zero, beginning from $n \geq 3$. If the conditions $E_{13}^{(n,l)} = 0 (n = 1, 2)$ are rewritten in terms of displacements, ordinary differential equations with respect to the variable η for calculating the fields $u^{(n,l)}$ are obtained. After they are integrated, the explicit dependence of the displacements on the fast variable η can be isolated, and arbitrary functions of the slow variables $\tilde{u}^{(n,l)}(X, T)$ emerge:

$$u^{(1,1)} = -\varepsilon ik\eta\tilde{w}^{(0,1)} + \tilde{u}^{(1,1)}, \quad u^{(1,l)} = \tilde{u}^{(1,l)}, \quad l = 0, \pm 2, \pm 3, \dots$$
$$u^{(2,1)} = -\varepsilon^2\eta\partial_X\tilde{w}^{(0,1)} + \tilde{u}^{(2,1)}, \quad u^{(2,l)} = \tilde{u}^{(2,l)}, \quad l = \pm 2, \pm 3, \dots$$

Here and below, all functions that do not depend on η are marked with a tilde, for example, $\tilde{u}^{(n,l)}(X, T)$.

From the relations $(P^{33})^{(m,l)} = 0$ we can find the relation between the components of the strain tensor, which is useful for further calculations

$$E_{33}^{(m,l)} = -\frac{\lambda}{\lambda + 2\mu} E_{\gamma\gamma}^{(m,l)}, \quad \gamma = 1, 2, \quad m = 2, 3 \tag{2.10}$$

In particular, equalities (2.10) lead to the representation

$$(P^{\alpha\beta})^{(m,l)} = \lambda' E_{\gamma\gamma}^{(m,l)}\delta_{\alpha\beta} + 2\mu E_{\alpha\beta}^{(m,l)}, \quad m = 2, 3, \quad \alpha, \beta, \gamma = 1, 2; \quad \lambda' = 2\lambda\mu/(\lambda + 2\mu)$$

which enables us to express the components $(P^{\alpha\beta})^{(m,l)}$ in terms of the functions $u^{(n,l)}$, $w^{(n,l)}$ $(n = 1, 2)$. Here λ' is the effective modulus of elasticity; in the linear approximation it specifies the stresses that appear when an area element of the shell is altered.

The scheme of the subsequent calculations can be illustrated by the example of the boundary-value problem

$$\partial_\eta (P^{13})^{(3,l)} + \varepsilon ikl (P^{11})^{(2,l)} = 0, \quad (P^{13})^{(3,l)}\big|_{\eta=\pm 1/2} = 0 \tag{2.11}$$

For the components $(P^{11})^{(2,l)}$ the preceding orders of perturbation theory reveal the dependence on the fast variable η and on the arbitrary functions $\tilde{u}^{(n,l)}$, but the functions $(P^{13})^{(3,l)}$ are still completely unknown. If Eq. (2.11) is integrated over the shell thickness and the boundary conditions are taken into account, we obtain the equality

$$\langle P^{11}\rangle^{(2,l)} = 0 \tag{2.12}$$

Here and below, $\langle f\rangle$ denotes the mean value of the function $f(\eta)$ across the shell thickness:

$$\langle f\rangle = \int_{-1/2}^{1/2} f(\eta)d\eta$$

The equality (2.12) is also the condition for boundary-value problem (2.11) to be solvable. In the present case it gives the algebraic relations between the functions already introduced

$$\varepsilon ik\tilde{u}^{(1,1)} = -\frac{\lambda}{\lambda + 2\mu}\varepsilon_1\tilde{w}^{(0,1)}, \quad \tilde{u}^{(1,2)} = -\frac{ik\varepsilon}{4}(w^{(0,1)})^2, \quad \tilde{u}^{(1,l)} = 0, \quad l = \pm 3, \pm 4, \dots$$

which mean that only two of the coefficients $(P^{11})^{(2,l)}$ are non-zero:

$$(P^{(11)})^{(2,1)} = (\varepsilon k)^2\eta(\lambda' + 2\mu)\tilde{w}^{(0,1)}$$

$$(P^{11})^{(2,0)} = (\lambda' + 2\mu)\{\varepsilon^2\partial_X^2\tilde{u}^{(0,0)} + (\varepsilon k)^2|\tilde{w}^{(0,1)}|^2\} + \lambda'\varepsilon_1\tilde{w}^{(0,0)}$$

Here only the components with $l \geq 0$ are presented, since the components with $l < 0$ can be found from the conditions for (2.3) to be real. The component $\tilde{w}^{(0,0)}$ will be determined by the next orders of perturbation theory. It is found that

$$\varepsilon_1\tilde{w}^{(0,0)} = -\frac{\lambda'}{\lambda' + 2\mu}\{\varepsilon^2\partial_X\tilde{u}^{(0,0)} + (\varepsilon k)^2|\tilde{w}^{(0,1)}|^2\} = -\frac{\lambda'}{\lambda' + 2\mu}E_{11}^{(2,0)} \tag{2.13}$$

Therefore,

$$(P^{11})^{(2,0)} = \frac{4\mu(\lambda' + \mu)}{\lambda' + 2\mu}E_{11}^{(2,0)} \tag{2.14}$$

When the solvability conditions are taken into account, the solutions of (2.11) have the form

$$(P^{13})^{(3,1)} = -\frac{1}{2}(\lambda' + 2\mu)ik^3\tilde{w}^{(0,1)}\left(\eta^2 - \frac{1}{4}\right)\varepsilon^3, \quad (P^{13})^{(3,l)} = 0, \quad l = 0, \pm 2, \pm 3, \ldots \tag{2.15}$$

The constraints imposed on the functions of the small variables (X, T) that were found enable us to return to relations (2.10). In terms of displacement fields, relations (2.10) reduce to ordinary differential equations in the variable η. Integrating them, we obtain the corrections $w^{(2,l)}$:

$$\frac{a}{d}w^{(2,0)} = -\frac{\lambda'}{2\mu}[\varepsilon_1\tilde{w}^{(0,0)} + E_{11}^{(2,0)}] - (\varepsilon k)^2|\tilde{w}^{(0,1)}|^2\eta + \frac{a}{d}\tilde{w}^{(2,0)}$$

$$w^{(2,1)} = -\frac{\lambda'}{4\mu}(\varepsilon k\eta)^2\tilde{w}^{(0,1)} - \frac{\lambda'}{\lambda' + 2\mu}\varepsilon_1\tilde{w}^{(0,1)}\eta + \tilde{w}^{(2,1)}, \quad w^{(2,l)} = \tilde{w}^{(2,l)}, \quad l = \pm 3, \pm 4, \ldots$$

After integration, the arbitrary functions of the slow variables $\tilde{w}^{(2,l)}$ appear again.

The general calculation scheme will be self-consistent if the functions of the slow variables, which are arbitrary in the first orders of perturbation theory, ultimately combine to form blocks so that a closed system of partial differential equations, which specifies their evolution under assigned initial and boundary conditions with respect to the slow variables, is obtained. This system will be a simplified model of the dynamics of a non-linear elastic shell. The conditions for a series of boundary-value problems to be solvable along a normal to the surface of a deformed shell in some order of perturbation theory reduce to a closed system of partial differential equations for calculating the functions of the slow variables only when the slow variables **X** and T are chosen correctly. This is also the main criterion for substantiating the procedure and the model itself. In this study "closure" occurs in sixth-order perturbation theory and gives equations for the envelope of the axisymmetric bends in the shell. "Closure" in such a high expansion order is not by chance, and it reflects the fine balance between the effects of dispersion and non-linearity that are characteristic of the problem under consideration. We shall show how the conditions for the perturbation theory equations to be solvable can be used to find the wave number of the neutrally stable linear mode that is responsible for the formation of annular folds on the shell surface and the "upper" critical load at which the mode forms and then to construct a model of the dynamics of bends in the shell near its stability threshold.

The conditions for the perturbation theory equations to be solvable indicate that a small number of the displacement fields $w^{(s,l)}$, $u^{(s+1,l)}$ $(s = 0, 2)$ must be known to construct the model. Apart from the components already calculated, only the harmonic $u^{(3,l)}$, which can easily be obtained by integrating Eq. (2.15), is required. It is much simpler to find the components $(P^{km})^{(s,l)}$ of the Piola–Kirchhoff tensor that are needed to construct the model and the derivatives $(\partial\varphi/\partial E_{km})^{(s,l)}$ associated with them from the perturbation theory equations than it is to calculate the displacement fields. This situation greatly reduces the number of calculations. The components $(P^{13})^{(s,l)}$ $((P^{11})^{(s,l)})$ of the Piola–Kirchhoff tensor must be calculated to the fifth (fourth) order inclusive. Note that the components $(P^{13})^{(n,l)}$ and $(P^{31})^{(n,l)}$ are not equal to one another, starting from $n = 3$, but according to the defining relation (1.3), they are related to one another, so that $(P^{31})^{(n,l)}$ can always be calculated from the known $(P^{13})^{(n,l)}$.

The fourth-order perturbation theory equations, which follow from the second equality in (2.1), have the form

$$\varepsilon ikl(P^{11})^{(3,l)} + \varepsilon^2\partial_X(P^{11})^{(2,l)} + \partial_\eta(P^{13})^{(4,l)} = 0, \quad (P^{13})^{(4,l)}\big|_{\eta=\pm1/2} = 0 \tag{2.16}$$

The preceding orders of perturbation theory specify all the functions in problem (2.16) except $(P^{13})^{(4,l)}$. The conditions for boundary-value problem (2.16) to be solvable yield not only algebraic relationships between the functions of the slow variables that appear in the integrations, but also the differential equation

$$\partial_X\langle P^{11}\rangle^{(2,0)} = 0 \tag{2.17}$$

Since the external stress at infinity $(\Theta^{11})^{(2,0)}$ is assumed to be constant, from (2.17) we have

$$\langle P^{11}\rangle^{(2,0)} \equiv (P^{11})^{(2,0)} = (\Theta^{11})^{(2,0)} = \text{const} \tag{2.18}$$

The functions $(P^{13})^{(4,l)}$ are calculated using the constraints found from (2.16). The only function that is not equal to zero is

$$(P^{13})^{(4,1)} = -i\varepsilon\partial_X\partial_k(P^{13})^{(3,1)} \tag{2.19}$$

This information about $(P^{13})^{(4,l)}$ is sufficient to construct the model. The relation (2.19) cannot be solved for the displacements $u^{(4,1)}$ because they do not appear in the simplified equation of the non-linear dynamics of a shell. The derivatives $(\partial\varphi/\partial E_{13})^{(4,1)}$, needed for the calculations, are found from (2.7) and (2.19). The form of the fields $w^{(3,l)}$ is also not needed to construct the effective equations. It is sufficient to have equations for the strains $E_{33}^{(3,l)}$, which are calculated using (2.10).

In fourth-order perturbation theory, from the first equality in (2.1) we obtain

$$\varepsilon ikl(P^{31})^{(3,l)} - \varepsilon_1(P^{22})^{(2,l)} + \partial_\eta(P^{33})^{(4,l)} = 0 \quad (P^{33})^{(4,l)}\big|_{\eta=\pm1/2} = 0 \tag{2.20}$$

The condition for boundary-value problem (2.20) to be solvable with respect to the variable η for $l=0$ gives the constraint (2.13). As a result, formula (2.14) is substantiated, and (2.18) can be rewritten in the form

$$(P^{11})^{(2,0)} = \frac{4\mu(\lambda'+\mu)}{\lambda'+2\mu}[\varepsilon^2\partial_X\tilde{u}^{(0,0)} + (\varepsilon k)^2|\tilde{w}^{(0,1)}|^2] = (\Theta^{11})^{(2,0)} = \text{const} \tag{2.21}$$

The condition for (2.20) to be solvable for $l=1$ gives the relation between the wave number of the neutrally stable mode, responsible for corrugation of the shell, and the external stress

$$-(\Theta^{11})^{(2,0)} = \frac{\lambda'+2\mu}{12}(\varepsilon k)^2 + \frac{4\mu(\lambda'+\mu)}{\lambda'+2\mu}\left(\frac{\varepsilon_1}{\varepsilon k}\right)^2$$

The condition for the function $[-(\Theta^{11})^{(2,0)}(k)]$ to have a minimum specifies the critical wave number k_0 and the critical load Θ:

$$(\varepsilon k_0)^2 = \frac{4\varepsilon_1}{\lambda'+2\mu}\sqrt{3\mu(\lambda'+\mu)}, \quad \Theta = -\frac{2d}{R}\sqrt{\frac{\mu(\lambda'+\mu)}{3}} = O(\varepsilon^2) \tag{2.22}$$

The expression for k_0 and Θ are identical to the expressions obtained previously[2] by analysing the linear Euler instability of a longitudinally compressed shell.

The conditions for the firth-order perturbation theory equations to be solvable provide algebraic relations between functions, which were arbitrary in the first orders of perturbation theory, and the equation

$$\partial_X\langle P^{11}\rangle^{(3,0)} = 0 \tag{2.23}$$

The further calculations are simplified somewhat when there are no longitudinal loads of order ε^3. Then Eq. (2.23) has the trivial solution $\langle P^{11}\rangle^{(3,0)} = 0$, which also relates the functions that appear as a result of the integrations. It is noteworthy that all the relations are self-consistent and not contradictory.

The conditions for the sixth-order perturbation theory equations to be solvable form a closed system of differential equations for calculating the fields $\sigma_{11}^{(4,0)}$, $\tilde{w}^{(0,1)}$, which comprise a simplified model of the non-linear elastic dynamics of the shell near its stability threshold:

$$\begin{aligned} \mu\varepsilon^4\partial_\tau^2\tilde{w}^{(0,1)} &= -k_0^2(\sigma_{11}^{(4,0)}-p)\tilde{w}^{(0,1)}+\frac{1}{3}\varepsilon^4k_0^2(\lambda'+2\mu)\partial_X^2\tilde{w}^{(0,1)}-q_2\varepsilon^4k_0^6\left|\tilde{w}^{(0,1)}\right|^2\tilde{w}^{(0,1)} \\ \mu\varepsilon^4k_0^2\partial_\tau^2\left|\tilde{w}^{(0,1)}\right| &= -\partial_X^2\{\sigma_{11}^{(4,0)}+q_1(\varepsilon k_0)^4\left|\tilde{w}^{(0,1)}\right|^2\} \end{aligned} \tag{2.24}$$

where

$$p = \varepsilon_1^2\left[\frac{3}{8}(\lambda'+2\mu)-\frac{\mu(\lambda'+\mu)}{30(\lambda'+2\mu)}+\frac{\mu(a_1+a_2)(3\lambda'+5\mu)}{3(\lambda'+\mu)(\lambda'+2\mu)}-\frac{a_1(\lambda'+2\mu)(\lambda'-\mu)}{6\mu(\lambda'+\mu)}+\frac{\lambda'^2}{16\mu}\right]$$

$$q_1 = \frac{\lambda'+2\mu}{4}\left(1-\frac{\lambda'}{2\mu}\right)+s,\quad q_2 = \frac{\lambda'+2\mu}{2}\left(\frac{1}{3}-\frac{\lambda'}{4\mu}\right)+s$$

$$s = \frac{3a_1+a_2}{6}+\frac{\lambda'^2a_1+2\mu^2(a_1+a_2)}{12\mu(\lambda'+2\mu)}$$

The parameters $a_{1,2}$ were previously introduced in Ref. 8 and have the meaning of effective moduli that characterize the “quasi-planar” non-linear elastic stress state of the shell:

$$a_1 = \frac{A}{2}+\frac{2\mu B}{\lambda+2\mu},\quad a_2 = \frac{1}{(\lambda+2\mu)^3}[-\lambda^3A+6\lambda^2\mu B+8\mu^3C]-\frac{A}{2}$$

Note that the calculation of the effective moduli of elasticity of the shell required a consideration of higher invariants in the expansion of the energy of the shell. Thus, in this problem, the development of local instabilities is caused not only by the geometric non-linearity of the medium, but also by its physical non-linearity and, therefore, cannot be investigated within the conventional theory of shells.

The quantity $\sigma_{11}^{(4,0)}$ is specified by the relation

$$\left(\frac{\partial\varphi}{\partial E_{11}}\right)^{(4,0)} = \sigma_{11}^{(4,0)}+\frac{\varepsilon_1^2}{6\mu(\lambda'+\mu)}[2\mu^2(a_1+a_2)+a_1(\lambda'+2\mu)^2]+\left|\tilde{w}^{(0,1)}\right|^2f(\eta) \tag{2.25}$$

where $f(\eta)$ is a definite function of η. When the shell radius becomes infinte, $\varepsilon_1\to 0$. Therefore, $\sigma_{11}^{(4,0)}$ describes a “quasi-planar” stress state of the shell that is uniform across its thickness.

Different solutions of system (2.24) correspond to different initial and boundary conditions with respect to the slow variables X and T.

When the shell is sufficiently long and the strains are uniform at its ends, the solution of the simplified model should be sought with the following boundary conditions

$$\begin{aligned} \tilde{w}^{(0,1)}\big|_{|x|\to\infty} &= \partial_\tau\tilde{u}^{(0,0)}\big|_{|x|\to\infty} = 0 \\ \sigma_{11}^{(4,0)}\big|_{|x|\to\infty} &= [\Theta^{11}]^{(4,0)}-\frac{1}{2\mu}\Theta^2\left\{1+\frac{3}{4\mu(\lambda'+\mu)}[2\mu^2(a_1+a_2)+a_1(\lambda'+2\mu)^2]\right\} \end{aligned} \tag{2.26}$$

The simplest way to obtain the expression presented for $\sigma_{11}^{(4,0)}\Big|_{|x|\to\infty}$ is as follows. Let a constant stress Θ^{ij} be applied to the shell at infinity. The unit non-zero component of this stress satisfies the condition

$$\Theta^{11} = (\Theta^{11})^{(2)}+(\Theta^{11})^{(4)}\ldots$$

The superscripts indicate the orders of the terms with respect to the parameter ε, and $(\Theta^{11})^{(2)}=\Theta$.

The uniform axisymmetric strains of the shell are described by the formulae

$$\upsilon^3 = y^3\tilde{w}, \quad \upsilon^1 = y^1\tilde{u}, \quad \upsilon^2 = 0$$

Dimensionless variables are used here. The non-zero components of the tensors $\nabla_s \upsilon^m$ and E_{sm} have the form

$$\nabla_1\upsilon^1 = \tilde{u}, \quad \nabla_2\upsilon^2 = \nabla_3\upsilon^3 = \tilde{w}$$

$$E_{11} = \tilde{u} + \frac{1}{2}\tilde{u}^2, \quad E_{22} = \left(\frac{y^3}{R}\right)^2\tilde{E}_{22}, \quad E_{33} = \tilde{E}_{22} = \tilde{w} + \frac{1}{2}\tilde{w}^2$$

For uniform strains the invariants of the tensor E_{km} have simple expressions

$$I_1 = E_{11} + \tilde{E}_{22} + E_{33}, \quad I_2 = E_{11}^2 + \tilde{E}_{22}^2 + E_{33}^2, \quad I_3 = E_{11}^3 + \tilde{E}_{22}^3 + E_{33}^3$$

When $|x| \to \infty$, the boundary conditions (1.4) reduce to the algebraic relations

$$P^{11} = (1+\tilde{u})\frac{\partial\varphi}{\partial E_{11}} = (1+\tilde{w})\Theta^{11}, \quad P^{33} = (1+\tilde{w})\frac{\partial\varphi}{\partial E_{33}} = 0 \tag{2.27}$$

The solutions of Eq. (2.27) were found by the method of successive approximations in the form of the expansions

$$\tilde{u} = \tilde{u}^{(2)} + \ldots, \quad \tilde{w} = \tilde{w}^{(2)} + \ldots, \quad E_{11} = E_{11}^{(2)} + E_{11}^{(4)} + \ldots, \quad E_{33} = E_{33}^{(2)} + E_{33}^{(4)} + \ldots \tag{2.28}$$

Relations (2.27) and (2.28) were first used to calculate second-order corrections with respect to ε and then to obtain the expression for the derivative

$$\left(\frac{\partial\varphi}{\partial E_{11}}\right)^{(4)} = [\Theta^{11}]^{(4)} - \frac{1}{2\mu}\Theta^2 \tag{2.29}$$

Comparing (2.29) and (2.25) and taking into account the equality (28), we obtain expression (2.26) for $\sigma_{11}^{(4,0)}\Big|_{|x|\to\infty}$.

3. Soliton-like excitations and structures

It is interesting that the system of Eqs. (2.21), (2.24), which describes the non-linear elastic dynamics of axisymmetric bends in the shell near its stability threshold is formally equivalent to the system of equations obtained in Ref. 14 when analysing the entirely different physical problem of the corrugation of the separate most strongly loaded layer of a material in a laminated medium. This indicates the universality of the simplified non-linear equations and enables us to use the exact solutions that were previously found and analysed in Ref. 14.

The non-linear model (2.21), (2.24) allows of solutions of the following form:

$$\tilde{u}^{(0,0)} = \frac{(\lambda' + 2\mu)}{4\mu(\lambda+\mu)\varepsilon^2}\Theta X - k_0^2\int^{X+VT} A^2(X')\,dX'$$

$$\tilde{w}^{(0,1)} = A(X+VT)\exp\{i(\Omega T + \kappa X + \varphi_0)\}$$

$$\sigma_{11}^{(4,0)} = c^{(4)} - A^2\varepsilon^4 k_0^2\{q_1k_0^2 + \mu V^2\}, \quad A^2 = |\tilde{w}^{(0,1)}|^2$$

where V, κ, Ω and φ_0 are real parameters that are related to one another:

$$\kappa = \frac{V\Omega}{V_{cr}^2}, \quad V_{cr}^2 = \frac{\lambda' + 2\mu}{3\mu}k_0^2$$

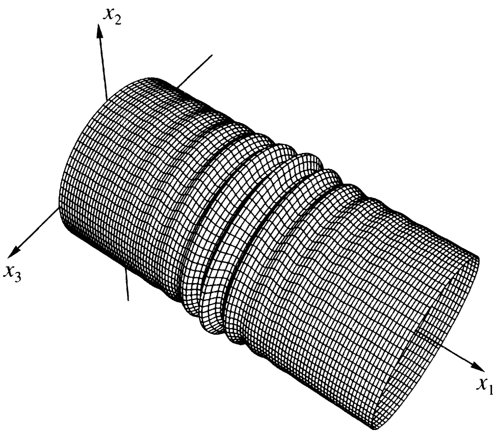

Fig. 1.

The integration constant is determined by the boundary conditions. In particular, when the strains in the shell are uniform at infinity, the constant is specified by conditions (2.26):

$$c^{(4)} = \sigma_{11}^{(4,0)}\Big|_{|x| \to \infty}$$

The function $A(X)$ is a solution of the ordinary differential equation

$$(\partial_X A)^2 = \alpha A^2 + \frac{\beta}{2}A^4 + c \tag{3.1}$$

where

$$\alpha = -\frac{k_0^2}{\mu\varepsilon^4(V^2 - V_{\rm cr}^2)}\left[c^{(4)} - p + \mu\varepsilon^4\left(\frac{\Omega}{k_0 V_{\rm cr}}\right)^2(V^2 - V_{\rm cr}^2)\right]$$

$$\beta = \frac{k_0^4}{\mu(V^2 - V_{\rm cr}^2)}[\mu V^2 + k_0^2(q_1 - q_2)]$$

and c is the integration constant.

Spatially localized, non-linear elastic folds form on the shell surface if the dimensions and material parameters of the shell, as well the physical parameters of the folds, satisfy the conditions

$$c = 0, \quad \alpha > 0, \quad \beta < 0$$

Such folds are grouped into “light” solitons

$$A = \frac{\sqrt{2\alpha/|\beta|}}{\operatorname{ch}(\sqrt{\alpha}[X + VT])} \tag{3.2}$$

According to (3.2), the axisymmetric bulges of the shell are localized along the generatrix in a region with the characteristic dimension $\alpha^{-1/2}$, which moves with velocity V (the annular folds can also be stationary). When $\Omega = \kappa = 0$, the transverse displacements of the shell surface have the form (Fig. 1)

$$w = \frac{2\sqrt{2\alpha/|\beta|}\cos(k_0\xi + \varphi_0)}{\operatorname{ch}(\sqrt{\alpha}[X + VT])} \tag{3.3}$$

The wave-like bends in the shell surface and their “subdivision” into the corrugation solitons (3.3) begin at some critical load, which is specified by the condition $\alpha > 0$ and differs in absolute value from $|\Theta|$ $(\Theta < 0)$. A shell with

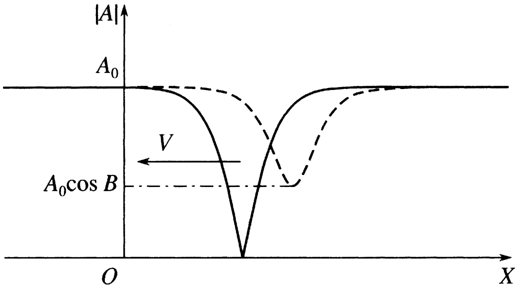

Fig. 2.

stationary soliton-like folds supports the load $|\Theta+[\Theta^{11}]^{(4,0)}|$, which is smaller than the upper critical load of the linear theory $|\Theta|$:

$$[\Theta^{11}]^{(4,0)} > \frac{1}{2\mu}[\Theta]^2\left\{1+\frac{3}{4\mu(\lambda'+\mu)}[2\mu^2(a_1+a_2)+a_1(\lambda'+2\mu)^2]+p\right\}$$

The difference between the external load and the value of Θ cannot be excessively large. Otherwise, the starting assumption (2.4) of perturbation theory is violated.

The formation of spatially localized annular folds on the surface of a longitudinally compressed shell has been observed experimentally. The presence of a pressure within the shell facilitates their formation.[2]

For $c>0$, $0>c>-\alpha^2/2|\beta|$, $\alpha>0$, $\beta<0$, there are two types of chains of light corrugation solitons, which differ with respect to the shape of the connecting piece that joins adjacent solitons in the chain.[14]

The light solitons and chains of these solitons can be regarded as precursors of the subsequent plastic flow of the material.

When $\alpha<0$ and $\beta>0$, bounded solutions are only possible for $0<c\le\alpha^2/(2\beta)$. They describe localized excitations on a background of a cnoidal elastic wave travelling along the generatrix of the shell. The simplest of these solutions is obtained for $c=\alpha^2/(2\beta)$ and represents a "dark" soliton:

$$w = A_0\,\mathrm{th}(\sqrt{|\alpha|/2}[X+VT])\cos(k_0\xi+\kappa X+\Omega T+\varphi_0),\quad A_0 = 2\sqrt{|\alpha|/\beta} \tag{3.4}$$

The solution (3.4) exhibits asymptotic behaviour at spatial infinity that resembles a transverse monochromatic elastic wave:

$$w\approx\pm A_0\cos(k_0\xi+\kappa X+\Omega T+\varphi_0)$$

The amplitude of the non-linear wave (3.4) has a spatially localized dip, i.e., a "trough" with the characteristic dimension $\alpha^{-1/2}$, which moves similar to a particle with velocity V (Fig. 2, the solid curve). In the region of this "trough," the amplitude of the non-linear wave falls to zero.

The solutions found for the model include solutions of the following form

$$\tilde{w}^{(0,1)} = A(X+VT)\exp\{i\omega T+iF(X+VT)\} \tag{3.5}$$

where ω and V are real parameters, and $A(X)$ and $F(X)$ are real functions. The solutions (3.5) correspond to non-linear waves that propagate along the generatrix of the shell on a background of annular folds on its surface. The traverse displacements of the shell have the form

$$w = 2A(X+VT)\cos[k_0\xi+\omega T+F(X+VT)] \tag{3.6}$$

By virtue of the conservation law

$$\partial_T\{w^*\partial_T w-\text{c.c.}\} = V_{\text{cr}}^2\partial_X\{w^*\partial_X w-\text{c.c.}\}$$

the functions A and F are related by the equality

$$G = A^2\{\omega V + (V^2 - V_{cr}^2\partial_X F)\} \tag{3.7}$$

Here and below, the constant G is assumed to be non-zero (when $G = 0$, solutions that were already considered are obtained).

Relation (3.7) can be used to reduce system (2.24) to an equation for calculating the function $z = A^2$:

$$(\partial_X z)^2/8 = Ez - \upsilon_0 - \upsilon_2 z^2 - \upsilon_3 z^3 \equiv f(z) \tag{3.8}$$

where

$$\upsilon_0 = \frac{1}{2}\left[\frac{G}{V^2 - V_{cr}^2}\right]^2, \quad \upsilon_2 = \frac{1}{2(V^2 - V_{cr}^2)}\left\{\omega^2 V_{cr}^2 + \frac{k_0^2(V^2 - V_{cr}^2)}{\mu\varepsilon^4}[c^{(4)} - p]\right\}, \quad \upsilon_3 = -\frac{\beta}{4}$$

and E is a new integration constant.

Bounded solutions of Eq. (3.8) exist only if there are real values of the roots z_1, z_2 and z_3 of the polynomial $f(z)$

$$4\upsilon_2^3\upsilon_0 + 27\upsilon_0^2\upsilon_3^2 + 2E\upsilon_3[9\upsilon_2\upsilon_0 - 2E^2] - \upsilon_2^2 E^2 < 0$$

and the following inequalities

$$0 < z_1 < z_2 \le z_3 \ (\upsilon_3 < 0, z_1 \le z \le z_2) \tag{3.9}$$

$$z_1 < 0 < z_2 < z_3 \ (\upsilon_3 > 0, z_2 \le z \le z_3) \tag{3.10}$$

The general solution of Eqs. (3.7) and (3.8) is written down in terms of Jacobi elliptic functions and incomplete elliptic integrals of the third kind.[15] The case (3.9) is observed for $\upsilon_2 > 0$, $\upsilon_3 > 0$. When the roots z_2 and z_3 are identical, the solution is expressed in terms of elementary functions:

$$\begin{aligned} A^2 &= A_0^2[1 - \sin^2 B\,\mathrm{sech}^2\Psi], \quad \Psi = (X + VT)\sqrt{2|\upsilon_3|}A_0\sin B \\ F &= \mathrm{arctg}(\mathrm{tg}B\,\mathrm{th}\Psi) + (X + VT)\left[\sqrt{2|\upsilon_3|}A_0\cos B - \frac{\omega V}{V^2 - V_{cr}^2}\right] \end{aligned} \tag{3.11}$$

Here, instead of the parameter E, which satisfies the inequalities

$$\frac{\upsilon_2^2}{3|\upsilon_3|} < E < \frac{\upsilon_2^2}{4|\upsilon_3|}$$

we have introduced the angular parameter B: $\sin^2 B = (z_2 - z_1)/z_2$. As a result, we obtain the relations

$$z_1 = \frac{\upsilon_2 - 2\sqrt{D}}{3|\upsilon_3|} = \frac{\upsilon_2\cos^2 B}{|\upsilon_3|(2 + \cos^2 B)}, \quad z_2 = A_0^2 = \frac{\upsilon_2 + \sqrt{D}}{3|\upsilon_3|} = \frac{\upsilon_2}{|\upsilon_3|(2 + \cos^2 B)}$$

$$\upsilon_0 = A_0^6|\upsilon_3|\cos^2 B, \quad \upsilon_2 = A_0^2|\upsilon_3|(2 + \cos^2 B), \quad \upsilon_3 = -\beta/4$$

$$\sqrt{D} = \upsilon_2^2 - 3|\upsilon_3|E = \frac{\upsilon_2\sin^2 B}{2 + \cos^2 B}, \quad E = A_0^4|\upsilon_3|(1 + 2\cos^2 B)$$

The solution (3.6), (3.11) describes a new type of "dark" soliton, namely, a "grey" soliton, on a background of a transverse monochromatic elastic wave propagating along the generatrix of the shell. A grey soliton represents particle-like formations in the form of a "trough" in the amplitude of a transverse non-linear wave. Unlike the dark soliton (3.4), the grey soliton (3.6), (3.11) has the additional degree of freedom B, which controls the depth of the "trough" (Fig. 2, the dashed curve). Since dark and grey solitons form only near the stability threshold of the shell shape, studying them can be useful for revealing and diagnosing the pre-critical stress state of a shell.

4. The construction of a model of two-dimensional bends in a shell

Different neutrally stable deformation modes are responsible for the formation of the diamond-shaped dents or annular folds on the surface of a longitudinally compressed shell. Therefore, the theoretical description of the patterns of diamond-shaped dents on the shell surface calls for the construction of another model.

We will formulate the conditions under which the two-dimensional dynamics of the local bends in the shell near its stability threshold can be faithfully described within a simple non-linear model.

Let the order of magnitude of the stresses along the shell edges satisfy the estimate

$$\Theta^{11}/\mu = O(\varepsilon^2) + O(\varepsilon^3)$$

We note that along with the sharp bends in the shell, which lead to the formation of dents on its surface, there are smoother deformations that are associated with the localization processes of specific groups of dents. The dynamics of the "envelopes" of such groups will be described by the slow variables

$$X_\alpha = \varepsilon^{1/2}\xi_\alpha, \quad \alpha = 1, 2, \quad T = \varepsilon^{3/2}\tau$$

To construct the model, we will seek a solution of Eqs. (2.1) and (1.4) in the form

$$u = \varkappa^{(1)}\xi_1 + \sum_{n=0}^{\infty} u^{(2+n/2)}(\eta, \boldsymbol{\xi}, \mathbf{X}, T), \quad \upsilon = \sum_{n=0}^{\infty} \upsilon^{(2+n/2)}(\eta, \boldsymbol{\xi}, \mathbf{X}, T)$$

$$w = w^{(0,0)} + w^{(1)}(\boldsymbol{\xi}, \mathbf{X}, T) + \sum_{n=0}^{\infty} w^{(2+n/2)}(\eta, \boldsymbol{\xi}, \mathbf{X}, T)$$

The constants $w^{(0,0)}$ and $\varkappa^{(1)}$, which characterize the uniform deformation of the shell under the action of the static compressive stresses $(\Theta^{11})^{(2)}$, will be found during the calculations. We consider processes that are comparatively slow in time, for which the following estimate holds: $\partial_\tau w/w = O(\varepsilon^{3/2})$.

The changes in the scale transformations and estimates compared with the preceding problem are attributed to the appearance of new non-linear interactions and the allowance for other resonant deformation modes. The correctness of the choice of slow variables is ultimately confirmed by the "closure" of reductive perturbation theory in the simplified model, which allows of the possibility of a balance between the dispersion and non-linearity effects that are characteristic of the present problem.

The calculations in the first orders of perturbation theory, which are similar to those presented in the preceding section, yield the dependence of the displacement fields that are needed to construct the model on η:

$$\begin{aligned} &u^{(s)} = -\varepsilon\eta\partial_{\xi_1}\tilde{w}^{(s-1)} + \tilde{u}^{(s)}, \quad \upsilon^{(s)} = -\varepsilon\eta\partial_{\xi_2}\tilde{w}^{(s-1)} + \tilde{\upsilon}^{(s)}, \quad s = 2, 3 \\ &u^{(2+1/2)} = -\varepsilon^{3/2}\eta\partial_{X_1}\tilde{w}^{(1)} + \tilde{u}^{(2+1/2)}, \quad \upsilon^{(2+1/2)} = -\varepsilon^{3/2}\eta\partial_{X_2}\tilde{w}^{(1)} + \tilde{\upsilon}^{(2+1/2)} \\ &w^{(2)} = -\frac{\lambda\eta}{\lambda+2\mu}E^{(2)}_{\alpha\alpha} + \tilde{w}^{(2)} \end{aligned} \tag{4.1}$$

Using this dependence we find the components of the strain tensor $E^{(n)}_{\alpha\beta}$ for $n = 2, 2 + 1/2, 3$. Among them, only the following are non-zero

$$E^{(2)}_{11} = \varepsilon\varkappa^{(1)}, \quad E^{(2)}_{22} = \varepsilon_1 w^{(0,0)}, \quad E^{(3)}_{\alpha\beta} = -\varepsilon^2\eta\partial_{\xi_\alpha}\partial_{\xi_\beta}\tilde{w}^{(1)} + \tilde{E}^{(3)}_{\alpha\beta}$$

$$\tilde{E}^{(3)}_{11} = \varepsilon\partial_{\xi_1}\tilde{u}^{(2)}, \quad \tilde{E}^{(3)}_{22} = \varepsilon\partial_{\xi_2}\tilde{\upsilon}^{(2)} + \varepsilon_1\tilde{w}^{(1)}, \quad \tilde{E}^{(3)}_{12} = \frac{\varepsilon}{2}[\partial_{\xi_1}\tilde{\upsilon}^{(2)} + \partial_{\xi_2}\tilde{u}^{(2)}]$$

The components $(P^{\alpha\beta})^{(n)}$ are calculated from the formula

$$(P^{\alpha\beta})^{(n)} = (\partial\varphi/\partial E_{\alpha\beta})^{(n)} = -\lambda'\delta_{\alpha\beta}E^{(n)}_{\gamma\gamma} + 2\mu E^{(n)}_{\alpha\beta}, \quad n = 2, 2+1/2, 3$$

When $n = 2$, we have two non-zero components

$$(P^{11})^{(2)} = (\lambda' + 2\mu)\varepsilon\varkappa^{(1)} + \lambda'\varepsilon_1 w^{(0,0)} \quad (P^{22})^{(2)} = (\lambda' + 2\mu)\varepsilon_1 w^{(0,0)} + \lambda'\varepsilon\varkappa^{(1)} \tag{4.2}$$

The constants $\varkappa^{(1)}$ and $w^{(0,0)}$ will be specified by the next orders of perturbation theory.

To simplify the calculations, in the present problem we will assume there are no compressive stresses $O(\varepsilon^{2+1/2})$ on the shell edges. This assumption can be reconciled with the form of the expansion (2.2) if $w^{(2+1/2)} = \tilde{w}^{(2+1/2)}$. We then obtain the series of equalities

$$E^{(2+1/2)}_{33} = E^{(2+1/2)}_{\alpha\beta} = (P^{\alpha\beta})^{(2+1/2)} = (\partial\varphi/\partial E_{\alpha\beta})^{(2+1/2)} = 0$$

The special features of the quasi-two-dimensional calculations can be clarified by an example. In fourth-order perturbation theory, the last two equalities in (2.1) lead to the boundary-value problem

$$\partial_\eta (P^{\alpha 3})^{(4)} + \varepsilon\partial_{\xi_\beta}(P^{\alpha\beta})^{(3)} = 0, \quad (P^{\alpha 3})^{(4)}\big|_{\sigma'} = 0; \quad \alpha = 1, 2 \tag{4.3}$$

Its solvability condition

$$\partial_{\xi_\beta}\langle P^{\alpha\beta}\rangle^{(3)} = 0$$

is equivalent to the system of differential equations that relate the "quasi-planar" displacements of the shell $\tilde{u}^{(2)}$, $\tilde{\upsilon}^{(2)}$ to the transverse bends $\tilde{w}^{(1)}$:

$$\begin{Vmatrix} (\lambda' + 2\mu)\partial^2_{\xi_1} + \mu\partial^2_{\xi_2} & (\lambda' + \mu)\partial_{\xi_1}\partial_{\xi_2} \\ (\lambda' + \mu)\partial_{\xi_1}\partial_{\xi_2} & (\lambda' + 2\mu)\partial^2_{\xi_2} + \mu\partial^2_{\xi_1} \end{Vmatrix} \begin{Vmatrix} \tilde{u} \\ \tilde{\upsilon} \end{Vmatrix}^{(2)} = -\frac{\varepsilon_1}{\varepsilon}\begin{Vmatrix} \lambda'\partial_{\xi_1}\tilde{w}^{(1)} \\ (\lambda' + 2\mu)\partial_{\xi_2}\tilde{w}^{(1)} \end{Vmatrix} \tag{4.4}$$

On the shell edges there are static compressive stresses $O(\varepsilon^3)$, which cause the uniform strains

$$\tilde{u}^{(2)} \sim \varkappa^{(2)}\xi_1, \quad \tilde{w}^{(1)} \sim w^{(1,0)} = \text{const}$$

We separate them from the displacement fields:

$$\begin{aligned} \tilde{u}^{(2)} &= \varkappa^{(2)}\xi_1 + \bar{u}^{(2)}(\boldsymbol{\xi}, \mathbf{X}, T), \\ \tilde{\upsilon}^{(2)} &= \bar{\upsilon}^{(2)}(\boldsymbol{\xi}, \mathbf{X}, T) \\ \tilde{w}^{(1)} &= w^{(1,0)} + \bar{w}^{(1)}(\boldsymbol{\xi}, \mathbf{X}, T) \end{aligned} \tag{4.5}$$

We recall that the functions $\tilde{u}^{(2)}$, $\tilde{\upsilon}^{(2)}$, $\tilde{w}^{(1)}$ appeared after eliminating the dependence on η, i.e., the "fastest" of the coordinates, in the displacement fields. This dependence characterizes the non-uniformity of the strains along a normal to the shell surface. After additional separation of the uniform strains in accordance with (4.5), the coordinates ξ and $\mathbf{X}$ in the functions $\bar{u}^{(2)}$, $\bar{\upsilon}^{(2)}$, $\bar{w}^{(1)}$ describe different scale levels of the deformation. The ξ coordinates describe wave-like bends, which are associated with the formation of dents on the shell surface, and the $\mathbf{X}$ coordinates specify smooth deformations, which lead to spatial localization of some groups of dents.

To reveal the dependence of the functions $\bar{u}^{(2)}$, $\bar{\upsilon}^{(2)}$, $\bar{w}^{(1)}$ on the "fastest" of the remaining ξ coordinates, we will seek solutions of Eq. (4.4) in the form

$$\bar{u}^{(2)} \sim u^{(2,\mathbf{k})}(\mathbf{X}, T)\exp(i(\mathbf{k}\boldsymbol{\xi})), \quad \bar{\upsilon}^{(2)} \sim \upsilon^{(2,\mathbf{k})}(\mathbf{X}, T)\exp(i(\mathbf{k}\boldsymbol{\xi}))$$

$$\bar{w}^{(1,\mathbf{k})} \sim w^{(2,\mathbf{k})}(\mathbf{X}, T)\exp(i(\mathbf{k}\boldsymbol{\xi}))$$

It is then reduced to an algebraic system that relates the Fourier components of the displacement fields:

$$\hat{S}\varphi^{(2,\mathbf{k})} = \frac{\varepsilon_1}{\varepsilon} i w^{(1,\mathbf{k})} f \tag{4.6}$$

where

$$\varphi^{(2,\mathbf{k})} = \left\| \begin{matrix} u^{(2,\mathbf{k})} \\ \upsilon^{(2,\mathbf{k})} \end{matrix} \right\|, \quad f = \left\| \begin{matrix} \lambda' k_1 \\ (\lambda' + 2\mu)k_2 \end{matrix} \right\|, \quad \hat{S} = |\mathbf{k}|^2\{\mu\hat{E} + (\lambda' + \mu)\hat{P}\}$$

$\hat{P}$ is a projection matrix with the elements $P_{\alpha\beta} = k_\alpha k_\beta/|\mathbf{k}|^2$, and $\hat{E}$ is the identity matrix.

Using the inverse matrix

$$\hat{S}^{-1} = \frac{1}{|\mathbf{k}|^2(\lambda' + 2\mu)}\left\{\hat{E} + \frac{\lambda' + \mu}{\mu}\sigma_2\hat{P}\sigma_2\right\}$$

where σ_2 is the Pauli matrix, from Eq. (4.6) we express the components $u^{(2,\mathbf{k})}$ and $\upsilon^{(2,\mathbf{k})}$ in terms of $w^{(2,\mathbf{k})}$, and we then calculate the Fourier harmonics of the components of the Piola–Kirchhoff tensor

$$(P^{\alpha\beta})^{(3,\mathbf{k})} = \varepsilon^2\eta\hat{L}_{\alpha\beta}\overline{w}^{(1,\mathbf{k})} + \langle P^{\alpha\beta}\rangle^{(3,\mathbf{k})}, \quad \langle P^{\alpha\beta}\rangle^{(3,\mathbf{k})} = \varepsilon_{\alpha\gamma}k_\gamma\varepsilon_{\beta\rho}k_\rho\varepsilon_1 w^{(1,\mathbf{k})}N$$

$$\hat{L}_{\alpha\beta} = \lambda'\delta_{\alpha\beta}\mathbf{k}^2 + 2\mu k_\alpha k_\beta, \quad N = 4\mu(\lambda' + \mu)k_1^2/[|\mathbf{k}|^2(\lambda' + 2\mu)]$$

and its uniform components

$$(P^{11})^{(3,0)} = (\lambda' + 2\mu)\varepsilon\varkappa^{(2)} + \lambda'\varepsilon_1 w^{(1,0)} \quad (P^{22})^{(3,0)} = (\lambda' + 2\mu)\varepsilon_1 w^{(1,0)} + \lambda'\varepsilon\varkappa^{(2)} \tag{4.7}$$

where $\varepsilon_{\alpha\beta}$ is the absolutely antisymmetric unit tensor ($\varepsilon_{12} = 1$). The constants $\varkappa^{(2)}$ and $w^{(1,0)}$ will be fixed by the conditions for the next orders of perturbation theory to be solvable.

When the solvability conditions (4.4) hold, boundary-value problem (4.3) has the solution

$$(P^{\alpha 3})^{(4)} = \left(\frac{\partial\varphi}{\partial E_{\alpha 3}}\right)^{(4)} = \frac{\varepsilon^3}{2}(\lambda' + 2\mu)\left(\eta^2 - \frac{1}{4}\right)\partial^2_{\xi_\gamma}\partial_{\xi_\alpha}\overline{w}^{(1)} \tag{4.8}$$

We find the components $(P^{3\alpha})^{(4)}$ from (1.3) and (4.8):

$$(P^{3\alpha})^{(4)} = (P^{\alpha 3})^{(4)} + (P^{\alpha\beta})^{(2)}\varepsilon\partial_{\xi_\beta}\overline{w}^{(1)} \tag{4.9}$$

Formulae (4.8) and (4.9) enable us to calculate the Fourier harmonics of the function $(P^{3\alpha})^{(4)}$ that are needed to construct the model.

The further calculations are more tedious, but they are performed using a scheme similar to the scheme described in the preceding section. We shall dwell on the key points.

The conditions for the first equation of (2.1) to be solvable in fourth- and fifth-order perturbation theory reduce to the algebraic relations

$$\varepsilon_1 w^{(s,0)} = -\frac{\lambda'\varepsilon\varkappa^{(s+1)}}{\lambda' + 2\mu}, \quad s = 0, 1 \tag{4.10}$$

$$-\langle P^{11}\rangle^{(2)} = -(\Theta^{11})^{(2)} = \frac{\lambda' + 2\mu}{12}\varepsilon^2\Theta^2 + \left(\frac{\varepsilon_1}{\varepsilon}\right)^2\frac{4\mu(\lambda' + \mu)}{(\lambda' + 2\mu)\Theta^2}; \quad \Theta \equiv \Theta(\mathbf{k}) = |\mathbf{k}|^2/k_1 \tag{4.11}$$

According to (4.2) and (4.7), formula (4.10) is equivalent to the equalities $\langle P^{22}\rangle^{(n,0)}=0$ and can be used to relate the longitudinal strains of the shell to the external stress:

$$\langle P^{11}\rangle^{(n,0)} = (\Theta^{11})^n = \frac{4\mu(\lambda'+\mu)}{\lambda'+2\mu}\varepsilon\varkappa^{(n-1)}, \quad n = 2, 3$$

The condition for a minimum of (4.11) with respect to Θ specifies the upper critical load

$$-q^{(2)} = 2\varepsilon_1\sqrt{\frac{\mu(\lambda'+\mu)}{3}} = -\frac{4\mu(\lambda'+\mu)}{\lambda'+2\mu}\varepsilon\varkappa^{(1)} \tag{4.12}$$

which is equivalent to the upper critical load known from an analysis of the linear Euler instability of a longitudinally compressed shell.[2]

The critical load (4.12) corresponds to the value

$$\Theta(\mathbf{k}) = \frac{|\mathbf{k}|^2}{k_1} = \frac{4\varepsilon_1\sqrt{3\mu(\lambda'+\mu)}}{\varepsilon^2(\lambda'+2\mu)} \equiv r_0^2$$

It is noteworthy that the function $\Theta(\mathbf{k})$ hardly varies when $\partial_{k_1}\Theta \approx 0$, $\partial_{k_2}\Theta \approx 0$. The conditions just enumerated yield the wave vectors of three neutrally stable deformation modes

$$\mathbf{k}_1 = \frac{r_0}{2}(1, 1), \quad \mathbf{k}_2 = \frac{r_0}{2}(1, -1), \quad \mathbf{k}_3 = r_0(1, 0) \tag{4.13}$$

The three neutrally stable resonant modes (4.13) primarily cause two-dimensional bends in the shell. Therefore, we take

$$\begin{aligned}\bar{w}^{(1)}(\boldsymbol{\xi}, \mathbf{X}, T) &= \varphi_1(\mathbf{X}, T)\exp(i(\mathbf{k}_1\boldsymbol{\xi})) + \\ &+ \varphi_2(\mathbf{X}, T)\exp(i(\mathbf{k}_2\boldsymbol{\xi})) + \chi(\mathbf{X}, T)\exp(i(\mathbf{k}_3\boldsymbol{\xi})) + \text{c.c.}\end{aligned} \tag{4.14}$$

where φ_1, φ_2 and χ are complex-valued functions of the slow variables $\mathbf{X}$ and T.

In the original coordinates y^1, y^2 the displacement fields should not vary as a result of the transformations $y^2 \to y^2 + 2\pi R$. This means that in dimensionless variables function (4.14) should have a period of $2\pi\varepsilon/\varepsilon_1$ along the coordinate ξ_2. Here and below, for simplicity we will assume that the geometrical dimensions of the shell are such that the requirement for function (4.14) to be periodic with respect to the coordinate ξ_2 is satisfied. The condition for function (4.14) to be periodic with respect to the slow variable X_2 will be discussed separately.

It is noteworthy that the bulk of the functions needed to construct the model are expressed in terms of quantities that have already been found:

$$\begin{aligned}
&\| \tilde{u}\ \tilde{v} \|^{(2+1/2)} = -i\varepsilon^{1/2}\hat{D}\| \tilde{u}\ \tilde{v} \|^{(2)} \\
&(P^{\alpha\beta})^{(3+1/2)} = \left(\frac{\partial\varphi}{\partial E_{\alpha\beta}}\right)^{(3+1/2)} = -i\varepsilon^{1/2}\hat{D}(P^{\alpha\beta})^{(3)} \\
&(P^{\alpha3})^{(4+1/2)} = \left(\frac{\partial\varphi}{\partial E_{\alpha3}}\right)^{(4+1/2)} = -i\varepsilon^{1/2}\hat{D}(P^{\alpha3})^{(4)} \\
&(P^{3\alpha})^{(4+1/2)} = -i\varepsilon^{1/2}\hat{D}(P^{3\alpha})^{(4+1/2)} \\
&(P^{\alpha3})^{(5)} = \left(\frac{\partial\varphi}{\partial E_{\alpha3}}\right)^{(5)} = -\frac{\varepsilon}{2}\hat{D}^2(P^{\alpha3})^{(4)} + \frac{1}{2}(\lambda'+2\mu)\varepsilon^3\left(\eta^2-\frac{1}{4}\right)\partial^2_{\xi_\gamma}\partial_{\xi_\alpha}\tilde{w}^{(2)} \\
&(P^{3\alpha})^{(5)} = (P^{\alpha3})^{(5)} + \varepsilon\partial_{\xi_1}\tilde{w}^{(2)}(P^{11})^{(2)} + \varepsilon\partial_{\xi_\gamma}\tilde{w}^{(2)}(P^{\alpha\gamma})^{(3)}
\end{aligned} \tag{4.15}$$

Here the differential operator $\hat{D} = \partial_{k_\gamma}\partial_{X_\gamma}$.

The simplified model of the nonlinear elastic dynamics of the shell follows from the condition for the boundary-value problem for sixth-order perturbation theory to be solvable. The necessary solvability condition is obtained by integrating the first equality in (2.1) over the thickness of the shell taking into account of the boundary conditions on its surface, and reduces to the equality

$$\mu\varepsilon^5\partial_T^2 w^{(1,\mathbf{k})} = i\varepsilon k_\alpha\langle P^{3\alpha}\rangle^{(5,\mathbf{k})} + \varepsilon^{3/2}\partial_{X_\alpha}\langle P^{3\alpha}\rangle^{(4+1/2,\mathbf{k})} - \varepsilon_1\langle P^{22}\rangle^{(4,\mathbf{k})} \quad (4.16)$$

The Fourier components $\langle P^{3\alpha}\rangle^{(n,\mathbf{k})}$ $(n=4+1/2, 5)$ on the right-hand side of (4.16) can be found from (4.15). The function $\langle P^{22}\rangle^{(4,\mathbf{k})}$ is expressed not only in terms of the previously calculated fields, but also in terms of the components $(\tilde{u}, \tilde{v})^{(3,\mathbf{k})}$, which are still unknown.

The equations for calculating the functions $(\tilde{u}, \tilde{v})^{(3,\mathbf{k})}$ follow from the conditions for the boundary-value problems for fifth-order perturbation theory to be solvable, which are related to the last two equalities of (2.1). The amount of computational work can be reduced considerably if it is noted that these conditions can be transformed into

$$-\varepsilon^2\hat{S}\left[\left\|\begin{matrix}\tilde{u}\\ \tilde{v}\end{matrix}\right\|^{(3,\mathbf{k})} + \frac{\varepsilon}{2}\hat{D}^2\left\|\begin{matrix}\tilde{u}\\ \tilde{v}\end{matrix}\right\|^{(2,\mathbf{k})}\right] + i\varepsilon\varepsilon_1\tilde{w}^{(2,\mathbf{k})}\left\|\begin{matrix}\lambda' k_1\\ (\lambda'+2\mu)k_2\end{matrix}\right\| + \left\|\begin{matrix}f_1\\ f_2\end{matrix}\right\|^{(5,\mathbf{k})} = 0$$

Here

$$f_\beta^{(5,\mathbf{k})} = i\frac{\varepsilon^3}{2}\left\{k_1\lambda'[(\partial_{\xi_1}\bar{w}^{(1)})^2 + (\partial_{\xi_2}\bar{w}^{(1)})^2]^{(2,\mathbf{k})} + 2\mu k_\alpha[\partial_{\xi_\alpha}\bar{w}^{(1)}\partial_{\xi_\beta}\bar{w}^{(1)}]^{(2,\mathbf{k})}\right\}$$

Here $[f\varphi]^{(n,\mathbf{k})}$ denotes the Fourier coefficient of order ε^n for the product of the functions $f(\xi)$ and $\varphi(\xi)$.

The solvability condition (4.16) ultimately yields a closed system of equations for the evolution of the "envelopes" of the three resonant modes, that are responsible for the non-linear elastic dynamics of bends the surface of the shell near its stability threshold:

$$\begin{aligned}
&(\partial_T^2 - s^2\partial_{X_2}^2)\varphi_1 + \frac{1}{4}\sigma\varphi_1 + g\chi\varphi_2^* = 0\\
&(\partial_T^2 - s^2\partial_{X_2}^2)\varphi_2 + \frac{1}{4}\sigma\varphi_2 + g\chi\varphi_1^* = 0\\
&(\partial_T^2 - s^2\partial_{X_1}^2)\chi + \sigma\chi + g\varphi_1\varphi_2 = 0
\end{aligned} \quad (4.17)$$

where

$$s^2 = \frac{\lambda'+2\mu}{3\mu}r_0^2,\quad \sigma = \frac{(\Theta^{11})^{(3)}}{\mu\varepsilon^3}r_0^2,\quad g = 3r_0^2\frac{\varepsilon_1(\lambda'+\mu)}{\varepsilon^3(\lambda'+2\mu)}$$

The absence of the function $\tilde{w}^{(2,\mathbf{k})}$, which belongs to the next order of perturbation theory, in Eq. (4.17) is ensured by the solvability condition (4.11) of the preceding order of perturbation theory.

Unlike the previously considered problem of the formation of annular folds on the surface of a longitudinally compressed shell, the theoretical description (in the principal approximation) of two-dimensional dent patterns requires only the first two terms in expansion (1.1) of the elastic energy. Hence, it may be concluded that the values of the effective moduli depend strongly on the dynamic symmetry of the problem in the simplified non-linear models.

The model Eq. (4.17) describe the evolution of the envelopes of dents on the shell surface. It can be shown that if an interaction of three groups of unstable deformation waves, which are near the resonant modes responsible for corrugating the shell surface, can be generated in the conventional equations of the non-linear theory of thin shells using reductive perturbation theory, these equations would be identical to the proposed simplified model. At the same time, the problems considered show that the development of local instabilities and the formation of dents on shell surfaces are not always specified by the interactions that are taken into account by the conventional models of shells.

From a mathematical point of view, system (4.17) describes three-wave "reactions" (see Fig. 3). First there is "merging" of the unstable deformation modes $\sim\exp(i(\mathbf{k}_\alpha\boldsymbol{\xi}))$ $(\alpha=1, 2)$ with wave vectors that are close to the values $\mathbf{k}_1 = r_0(1, 1)/2$ and $\mathbf{k}_2 = r_0(1, -1)/2$, which results in the appearance of wave-like bends in the shell along the generatrix.

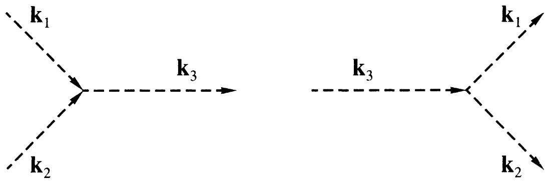

Fig. 3.

Second there is a transformation of the unstable deformation mode $\sim\exp(i(\mathbf{k}_3\boldsymbol{\xi}))$ ($\mathbf{k}_3 \simeq r_0(1, 0)$) into two waves, which are responsible for corrugating the shell in directions that make angles of $\pm 45°$ with the generatrix.

Apart from the traditional energy and momentum conservation laws, model (4.17) allows of special conservation laws of the Manley–Rowe type [13]:

$$\begin{aligned}&\partial_T(\varphi_\alpha^*\partial_T\varphi_\alpha + \chi^*\partial_T\chi - \text{c.c.}) - s^2\{\partial_{X_2}(\varphi_\alpha^*\partial_{X_2}\varphi_\alpha - \text{c.c.}) + \\ &+\partial_{X_1}(\chi^*\partial_{X_1}\chi - \text{c.c.})\} = 0, \quad \alpha = 1, 2\end{aligned} \tag{4.18}$$

In Eq. (4.18) there is no summation over the repeating subscript α.

5. Dent patterns and solitary waves on a shell surface

In interpreting system (4.17), special attention should be focused on the fact that the displacement fields of the shell are periodic functions of the original dimensional coordinate y^2. This means that only solutions of system of Eq. (4.17) with a period $\varepsilon^{3/2}2\pi/\varepsilon_1$ with respect to the slow coordinate X_2 have physical meaning.

The simplest of the solutions of system (4.17)

$$\varphi_1 = \varphi_2 = \mp 2\mu \equiv \varphi\left(X_1 + \frac{\gamma}{2}X_2\right) = \varphi^*\left(X_1 + \frac{\gamma}{2}X_2\right), \quad \gamma = \pm 1$$

describes stationary diamond-shaped dents on the shell surface. The transverse displacements of the shell surface have the form

$$\begin{aligned}\bar{w}^{(1)} &= \pm A(\Psi)\left\{\cos\frac{r_0\xi_1}{2}\cos\frac{r_0\xi_2}{2} - \cos r_0\xi_1\right\}\\ A(\Psi) &= \frac{(\lambda' + 2\mu)R}{\mu(\lambda' + \mu)d}(\Theta^{11})^{(3)}[\alpha_2 + (\alpha_3 - \alpha_2)\text{cn}^2(\Psi, k)]\\ \Psi &= \frac{2\varepsilon_1}{\pi\varepsilon^{3/2}}K(k)\left(X_1 + \frac{\gamma}{2}X_2\right) \equiv \frac{2K(k)}{R\pi}\left(y^1 + \frac{\gamma}{2}y^2\right)\end{aligned} \tag{5.1}$$

Here $A(\Psi)$ is the envelope of the dents on the shell surface, and the α_i ($i = 1, 2, 3$) are roots of the equation

$$\alpha^3 + \delta^2 - \alpha^2 = 0$$

The positive real parameter δ^2 should satisfy the constraint

$$0 \le \delta^2 < \frac{4}{27} \tag{5.2}$$

from which it follows that the α_i are real numbers. Their relative positions on the number axis correspond to the inequality

$$\alpha_1 \le 0 \le \alpha_2 < \alpha_3$$

The modulus k of the Jacobi function cn(Ψ, k), and the complete elliptic integral of the first kind $\mathbf{K}(k)$ is specified by the relations

$$k^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 + |\alpha_1|}, \quad \frac{\pi R}{4d}\left[\frac{3(\Theta^{11})^{(3)}(\alpha_3 + |\alpha_1|)}{\lambda' + 2\mu}\right]^{1/2} = \mathbf{K}(k)$$

In these formulae $(\Theta^{11})^{(3)} > 0$. Therefore, a shell with dents sustains an external stress whose absolute value is less than the upper critical load (4.12) given by the linear theory.

The displacement field (5.1) describes a band of diamond-shaped dents, which encircles the cylindrical shell along a helix with pitch πR. The centre line of the band makes an angle arctg $2 \approx 70°$ with the generatrix of the shell.

Let us evaluate the characteristic dimension l of the dents from the condition $r_0 \sim 1$, which is a necessary condition for self-consistency of the assumptions of perturbation theory. This gives

$$l \sim \sqrt{dR}, \quad \varepsilon \sim \sqrt{d/R}$$

According to the constraint (5.2), the parameter δ^2 is small. Therefore, the approximate value of the modulus of the elliptic functions can be found from the equation

$$K(k) \simeq \frac{\pi R}{4d}\sqrt{\frac{3(\Theta^{11})^{(3)}}{\lambda' + 2\mu}}$$

The envelope A in the direction perpendicular to the band of dents can be approximated quite well by the function

$$A \simeq \frac{(\lambda' + 2\mu)R}{\mu(\lambda' + \mu)d}(\Theta^{11})^{(3)}\operatorname{sech}^2\left(\frac{1}{4}\sqrt{\frac{15(\Theta^{11})^{(3)}}{\varepsilon^3(\lambda' + 2\mu)}}\frac{\varepsilon^{3/2}}{d}y\right)$$

where y is the dimensional coordinate in the direction indicated. Hence we obtain the estimate of the characteristic localization scale of the dents

$$d\varepsilon^{-3/2} \sim (d/R)^{1/4}R \sim l\varepsilon^{-1/2}$$

For $d/R \sim 0.01$, there can be about three dents within the localization scale of the bends. A shell with dents then sustains the compressive stress $-q^{(2)} + (\Theta^{11})^{(3)}$, whose absolute value is tens of percent less than the upper critical load $q^{(2)}$ given by the linear theory. The lower critical load that can be supported by a shell with dents can be found from an analysis of the stability of the solution (5.1) of system (4.17) with respect to small two-dimensional perturbations.

Fig. 4 shows a copy of a photograph of a longitudinally compressed shell with dents.[2] which closely agrees with the solution (5.1).

For special initial and boundary conditions with respect to the slow variables $\mathbf{X}$ and T, the formation of two-dimensional solitary waves on the surface of a longitudinally compressed shell that move along the generatrix is possible.

An extensive class of waves with bending of the shell surface is given by the substitution

$$\varphi_1 = \varphi_2 = f(X_1 + sT)\exp\left(i\frac{\varkappa}{2}X_2\right), \quad \chi = -\frac{g}{\sigma}f^2(X_1 + sT)\exp(i\varkappa X_2) \tag{5.3}$$

where $f(X_1)$ is a complex-valued function, $\varkappa = 2n\varepsilon_1/\varepsilon^{3/2}$, and n is an integer. We will express the function f in terms of the real amplitude A and the real phase Φ

$$f = A\exp i\Phi$$

Solutions of the type (5.3) correspond to the following transverse bends of the shell

$$\bar{w}^{(1)} = 4A\cos\left(\frac{r_0\xi_1}{2} + \Phi + \frac{\varkappa X_2}{2}\right)\cos\left(\frac{r_0\xi_2}{2}\right) - \frac{2g}{\sigma}A^2\cos(r_0\xi_1 + 2\Phi + \varkappa X_2) \tag{5.4}$$

Fig. 4.

From the conservation law (4.18) we find the relation between Φ and A

$$\partial_{X_1}\Phi = c_1/A^2 \tag{5.5}$$

where c_1 is an integration constant. When $c_1 = 0$, system (4.17) reduces to Eq. (3.1) for calculating A, where

$$\alpha = -(\sigma + s^2\varkappa^2)/(2s^2), \quad \beta = g^2/(\sigma s^2)$$

and c is another integration constant.

When $c_1 \neq 0$, system (4.17) is equivalent to Eq. (3.8) for determining $z = A^2$, where

$$\upsilon_0 = c_1^2/2, \quad \upsilon_2 = (\sigma + s^2\varkappa^2)/(8s^2), \quad \upsilon_3 = -g^2/(4\sigma s^2)$$

and E is a third integration constant.

It is interesting that Eqs. (3.1) and (3.8) appear in the analysis of different physical problems associated with the uniaxial loading of test pieces: in the description of the corrugation of a three-layer medium[14] and in the discussion of bends in a cylindrical shell. This is evidence of the latent dynamic symmetry of such problems. The possible types of solutions of the equations indicated were classified in the preceding section. In the case under discussion, such solutions correspond to the two-dimensional non-linear waves (5.4), among which the “light,” “dark” and “grey” solitons, which move like particles with a velocity s along the generatrix of the shell on a background of an array of diamond-shaped dents, are the most interesting.

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